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Orthogonal Matrix for Constructing Blocks of Centrosymmetric Matrices with Lower Hessenberg Block Structure

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Abstract

Efficient computation involving special structured matrices often benefits from sparse matrix representations. One such matrix, known for its central symmetry, is the centrosymmetric matrix. This class of matrices, which may feature lower Hessenberg matrices as blocks, arises in various applications in science and engineering. Given their unique structure, centrosymmetric matrices exhibit a range of distinctive properties. Through a review of these properties, this study constructs a block matrix using orthogonal matrices. The construction process is based on the classification of special orthogonal matrices, which differ depending on whether the matrix order is even or odd. This research develops orthogonal matrices for both even- and odd-ordered centrosymmetric matrices where the blocks follow a lower Hessenberg form. The resulting construction reduces matrix dimensions, contributing to more efficient computational performance.

Keywords: Centrosymmetric Matrix; Lower Hessenberg; Orthogonal Matrix; Matrix Construction; Symmetry; Sparse Representation

Abstrak

Perhitungan yang efisien terhadap matriks dengan struktur khusus seringkali memerlukan representasi dalam bentuk matriks jarang (*sparse matrix*). Salah satu jenis matriks yang memiliki simetri terhadap pusatnya adalah matriks sentrosimetris. Kelas matriks ini, yang dapat memiliki matriks Hessenberg bawah sebagai bloknya, muncul dalam berbagai aplikasi di bidang sains dan teknik. Berdasarkan struktur khasnya, matriks sentrosimetris memiliki beragam sifat khusus. Melalui tinjauan literatur terhadap sifat-sifat tersebut, studi ini membangun matriks blok dengan menggunakan matriks ortogonal. Proses konstruksi ini didasarkan pada klasifikasi matriks ortogonal khusus, yang dibedakan menurut ordo matriks yang genap atau ganjil. Penelitian ini mengembangkan matriks ortogonal untuk matriks sentrosimetris berordo genap dan ganjil, dimana blok-bloknya berbentuk Hessenberg bawah. Konstruksi yang dihasilkan mampu mereduksi ukuran matriks, sehingga memberikan kontribusi terhadap proses komputasi yang lebih efisien.

Kata Kunci: Matriks Sentrosimetris; Hessenberg Bawah; Matriks Ortogonal; Konstruksi Matriks; Simetri; *Sparse* Representasi

Introduction

Based on real-world applications, such as in control theory, vibration theory, molecular spectroscopy, the development of numerical methods, and the solution of ordinary and partial differential equations (Fletcher, 1983; Wonham, 1979; Barcilon, 1976; Joseph, 1992; Friedland, 1979; Aceto & Trigiante, 2007; Aceto & Trigiante, 2002) there is a growing need to evaluate the analytical processes involving a class of special matrices known as centrosymmetric matrices. Consequently, it is essential to develop analytical algorithms for computing centrosymmetric matrices at a fundamental level, prior to applying them in these various scientific and engineering domains.

Focusing on the structure of centrosymmetric matrices, the analytical process begins by computing their individual entries. A centrosymmetric matrix is a special type of matrix characterized by a unique structural property—symmetry about its center. This form of matrix has big size number of ordo at theory of information, linear system, and also at numerical analysis (Peng, Hu, & Zhang, 2004). The paper also focuses on elaborating of the necessary and sufficient condition for general solution on symmetric matrix having the submatrix.

The evaluation of the centrosymmetric matrix has been developed that sparse matrix arise as the block matrix, called lower Hessenberg form (Khasanah & Kuntarini, 2020). Given the importance of the lower Hessenberg structure in numerical approximation methods, it becomes necessary to evaluate centrosymmetric matrices that incorporate lower Hessenberg matrices as their blocks.

The entry structure of a centrosymmetric matrix with a lower Hessenberg block matrix (which exhibits a pattern similar to a butterfly structure) results in a sparse matrix form (Khasanah, Surarso, & Farikhin, 2020). This condition provides computational benefits in terms of complexity. The overall computational complexity of sparse matrices is lower than that of dense matrices. The operations on this special matrix are determined by the structure of its entries. A similar study was conducted by (Zhao & Li, 2015), whose findings indicate that the evaluation of the inverse and determinant of a special matrix can be performed more efficiently by applying sparse matrix computation methods, starting from the original dense form of the special matrix.

The construction of sparse matrices using orthogonal matrices serves as the central approach of this study. By utilizing the properties of orthogonal matrices, the structure of odd- and even-ordered centrosymmetric matrices (with lower Hessenberg matrices as blocks) can be transformed into a block-structured sparse matrix. Accordingly, this study proposes an evaluation of the construction of orthogonal matrices for both odd and even orders, aiming to produce centrosymmetric matrices with lower Hessenberg block structure in sparse form. This finding enables centrosymmetric matrices to be represented as sparse matrices, which is highly advantageous for improving computational efficiency.

Method

The method of this study is literature review from previous related books and scientific journals, especially those related to the Matrix Theory and Matrix Computation. This study focused on properties of the special matrix, called centrosymmetric matrix. Accordingly, it needs the fundamental theory on the definition and theorems on centrosymmetric matrix, lower Hessenberg matrix, and orthogonal matrix. Through the analytical process of matrix operations involving orthogonal matrices, the blocks of special matrices can be systematically constructed and evaluated, forming the basis for further discussion in this study. This finding construct block of centrosymmetric matrix by using their orthogonal matrices of odd and even matrix of centrosymmetric matrix. Such matrix structures are essential in various applied fields, where the use of orthogonal matrix rules plays a critical role in ensuring efficient and accurate computation.

Before presenting the main findings, we first outline the fundamental definitions, lemmas, and theorems that serve as the essential notations and theoretical foundation for the subsequent discussion on constructing orthogonal matrices for odd- and even-ordered centrosymmetric matrices, as presented below.

Definition 1. (Anton & Rorres, 2013) Let A be a $n \times n$ special matrix and x be a vector with nonzero in R^n. Therefore, the x is called an eigenvector of the matrix A corresponding to λ where

$$Ax = \lambda x$$

with for some scalar λ. In this case the λ can be called an eigenvalue of A.

Definition 2. (Khasanah & B. Surarso, 2018) Let $A = \left(a_{ij}\right)_{n\times n} \in R^{n\times n}$ be a special matrix called centrosymmetric matrix, where $a_{ij} = a_{n-i+1,n-j+1,}1 \le i \le n, 1 \le j \le n$ and A can be constructed as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{2,n} & \cdots & a_{22} & a_{21} \\ a_{1,n} & \cdots & a_{12} & a_{11} \end{pmatrix}.$$

Theorem 1. (Khasanah & Surarso, 2017) A matrix $A = \left(a_{ij}\right)_{n\times n} \in R^{n\times n}$ is a centrosymmetric matrix if and only if $J_n A J_n = A$ where

$$J_n = (e_n, e_{n-1}, \cdots, e_1) = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and e_i is the unit vector that has i −th entry equal to the number one and all other entries equal to number zero.

Proof. Let $n \times n$ centrosymmetric matrix with $n \times n$ of J_n matrix. Therefore,

$$J_n A J_n = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{2,n} & \cdots & a_{22} & a_{21} \\ a_{1,n} & \cdots & a_{12} & a_{11} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,n} & a_{1,n-1} & \cdots & a_{11} \\ a_{2,n} & a_{2,n-1} & \cdots & a_{21} \\ \vdots & \vdots & & \vdots \\ a_{21} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{11} & \cdots & a_{1,n-1} & a_{1,n} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{2,n} & \cdots & a_{22} & a_{21} \\ a_{1,n} & \cdots & a_{12} & a_{11} \end{pmatrix}$$

$$= A.$$

Lemma 2. Let $J_n \in R^{n\times n}$ be matrix with a nontrivial involution i.e. $J_n = J_n^{-1} \neq \pm I$ where $J_n^2 = I$, then $A \in R^{n\times n}$ is called J_n −symmetric if there is $J_n A J_n = A$.

Proof. Let J_n matrix from Theorem 1, then

$$J_n^{-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It can be seen that $J_n = J_n^{-1} \neq \pm I$.

The other side,

$$J_n J_n = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= I_n.$$

Results

In this section, we present a detailed analytical process for constructing an orthogonal matrix based on a centrosymmetric matrix with a lower Hessenberg block structure, developed from the following findings.

Lemma 3. (Z-Y. Liu, 2003) Let the A as the centrosymmetric matrix.

1. If n is even number as the size of matrix, then the matrices
   $A = \begin{pmatrix} B & JCJ \\ C & JBJ \end{pmatrix} \in V_n$ and $A = \begin{pmatrix} B - JC & 0 \\ 0 & B + JC \end{pmatrix}$
   are the matrix that figuring orthogonally similar, where there is the block matrix notate as B and C of centrosymmetric matrix.
2. If n represents the size number odd matrix, the matrices

   $$\begin{pmatrix} B & x & C^T \\ x^T & q & x^T J \\ C & Jx & JBJ \end{pmatrix} \in V_n \text{ and } \begin{pmatrix} B - JC & 0 & 0 \\ 0 & q & \sqrt{2}x^T \\ 0 & \sqrt{2x} & B + JC \end{pmatrix}$$

   also has the correlation of the matrix such orthogonally similar with the structure mentioned.

Proof.

1. Let the matrix written

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -J \\ I & J \end{pmatrix}$$

presents the orthogonal matrix for, and the multiplication to the A matrix gives the results such

$$QAQ^T = \begin{pmatrix} B - JC & 0 \\ 0 & B + JC \end{pmatrix}.$$

2. The matrix

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 & -J \\ 0 & \sqrt{2} & 0 \\ I & 0 & J \end{pmatrix}$$

is orthogonal odd centrosymmetric matrix, and the result follows when it operates to the A matrix.

The previous lemma can be written on the next theorem for the discussion of the rule of orthogonal matrix of its kinds of centrosymmetric matrix.

Theorem 4. (Golub & Van der Vorst, 2000)

1. Suppose n is even and

$$A = \begin{pmatrix} B & JCJ \\ C & JBJ \end{pmatrix} \in V_n$$

We may write $n/2$ represents skew symmetric which have orthonormal eigenvectors notated such v_i of A and correspondences to the eigenvalues λ_i from the operation of the equation can be evaluated such

$$(B - JC)u_i = \lambda_i u_i$$

where the number of $i \in I(n/2)$ and the eigenvector of this matrix is $v_i = (1/\sqrt{2})[u_i, -Ju_i]^T$, then the set with structure an orthonormal written on the notation u_i. By the same notations, the $n/2$ eigenvectors of symmetric orthonormal notated with w_i and corresponding eigenvalues p_i as eigenvalues that corresponding the matrix, then the solution of this case can be determine based on the equation

$$(B + JC)y_i = p_i y_i$$

with the number of $i \in I(n/2)$, and the notation of its eigenvector such $w_i = (1/\sqrt{2})[y_i, \boldsymbol{J}y_i]^T$, and the y_i, this structure is the orthonormal set. By to the discussion, set $\{v_1, v_2, \dots, v_n/2, w_1, w_2, \dots, w_n/2,\}$ of n size at eigenvectors of A matrix constructs the set with orthonormal structure, which therefore figures the eigenspace of the matrix A.

2. Let n is odd number of sizes of centrosymmetric matrix, and

$$A = \begin{pmatrix} B & x & JCJ \\ X^T & q & X^TJ \\ C & J_x & JBJ \end{pmatrix} \in V_n,$$

This part can be written by f the eigenvectors of skew symmetric orthonormal written on v_i of A matrix and correspondent to the eigenvalues λ_i from the answer of the equation

$$(B - JC)u_i = \lambda_i u_i$$

with the variable $i \in I(f)$ and the eigenvector $v_i = (1/\sqrt{2})[u_i, 0, Ju_i]^T$, and the u_i as form the set with orthonormal that related to the case of this process. Using the same steps, the symmetric-orthonormal with eigenvectors such w_i and the eigenvalues that is correspondence with this case can be written on p_i has the solution by analytical step such

$$\begin{pmatrix} A + JC & \sqrt{2}x \\ \sqrt{2}x^T & q \end{pmatrix} \begin{pmatrix} y_i \\ \alpha_i \end{pmatrix} = p_i \begin{pmatrix} y_i \\ \alpha_i \end{pmatrix}$$

where $i \in I(r)$ the eigenvector of $w_i = (1/\sqrt{2})[y_i, 2\alpha_i, Jy_i]^T$, also the matrix $\begin{pmatrix} y_i \\ \alpha_i \end{pmatrix}$ results the set of orthonormal. Then, the set $\{v_1, v_2, \dots v_f, w_1, w_2, \dots, w_r\}$ of eigenvector with size n from A constructs the set with orthonormal structure, that is spans the eigenspace for the matrix A.

Proof. Using previous lemma, we have

$$\begin{pmatrix} B - JC & 0 \\ 0 & B + JC \end{pmatrix}.$$

Then if $Z = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, this form of the Z matrix represent as orthogonal and that

$$Z^TQAQ^TZ = \begin{pmatrix} diag(\lambda_i) & \\ & diag(p_i) \end{pmatrix}.$$

Focusing also on the matrix Q^TZ is orthogonal, this results that the columns of the matrix operation of Q^TZ are the eigenvalues of A, matrix that is correspondence to the eigenvaluesλ_i and p_i. Moreover, the operation such

$$Q^TZ = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -J & J \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} U & V \\ -JU & JV \end{pmatrix}$$

which establishes the result.

On the first step, let written such $k = [n/2]$, and also identify the set of all centrosymmetric matrices as the result of orthogonal matrices.

When $n = 2k$ as the size of matrix with even number, let

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}$$

and when the size of matrix odd number such the size of matrix $n = 2k + 1$, then let

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}$$

It is clear that, Q has the rule of the matrix having an orthogonal matrix that appropriate for all of the number of n.

The second step, there is the $X \in R^{n\times m}$, notated the

$$Q^T X = \begin{pmatrix} X_{Q_1} \\ X_{Q_2} \end{pmatrix}, X_{Q_1} \in R^{(n-k)\times m}, X_{Q_2} \in R^{k\times m}.$$

Discussion

Based on the results above, centrosymmetric matrix having special structure (Liu, 2003; Z-Y. Liu, 2003; Anton & Rorres, 2013; Peng, Hu, & Zhang, 2004) as well as lower Hessenberg arise as block matrix presents two kind of its orthogonal (Golub & Van der Vorst, 2000; Khasanah & Kuntarini, 2020). The first is even orthogonal matrix having the structure such

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -J \\ I & J \end{pmatrix}$$

and odd orthogonal matrix with the structure

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 & -J \\ 0 & \sqrt{2} & 0 \\ I & 0 & J \end{pmatrix}.$$

Two kinds of those matrix are result of two centrosymmetric matrices classified based of its size. The two of these orthogonal matrices have the rule for constructing the centrosymmetric matrix to form sparse centrosymmetric matrix which perform lower computational process rather than dense centrosymmetric matrix (Zhao & Li, 2015; Khasanah & Kuntarini, 2020).

This finding supports the application process by introducing the centrosymmetric matrix as the initial step in computation, which requires a fundamental treatment prior to proceeding to the next phase. The subsequent computational steps can then be carried out more efficiently by applying orthogonal matrix operations (Barcilon, 1976; Aceto & Trigiante, 2002; Aceto & Trigiante, 2007; Anton & Rorres, 2013; Fletcher, 1983; Friedland, 1979; Joseph, 1992; Khasanah & Surarso, 2017). Moreover, orthogonal matrices can be applied to centrosymmetric matrices (particularly when these matrices are represented in sparse form) as demonstrated in the following examples.

Example 1. An even centrosymmetric matrix satisfies the following condition

$$A = JAJ$$

where J is the exchange matrix (anti-identity matrix). The matrix must also have a lower Hessenberg structure, meaning non-zero elements are on or below the first superdiagonal. Here is an example of an even centrosymmetric matrix A,

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix}$$

This matrix is centrosymmetric because it satisfies $A = JAJ$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It also has a lower Hessenberg structure because non-zero elements are on or below the first superdiagonal. An orthogonal matrix satisfies $Q^T Q = I$ where Q^T is the transpose of Q, and I is the identity matrix. Here is an example of an orthogonal matrix Q

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

This matrix is orthogonal because $Q^T Q = I$. We perform the matrix multiplication AQ,

$$AQ = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

The result is

$$AQ = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 & 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ 2 & 2 & -1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The transpose of matrix A is

$$A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since A is centrosymmetric and symmetric, $A = A^T$. Now we compute $Q^T \cdot A$

$$Q^T \cdot A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix}$$

The result is

$$Q^T \cdot A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

On the other side, for odd orthoonal matrix relating to the centrosymmetric matrix with lower Hessenberg as the block matrix can be illustrated as the following example.

Example 2. An odd centrosymmetric matrix satisfies the following condition

$$A = -JAJ$$

where J is the exchange matrix (anti-identity matrix). The matrix must also have a lower Hessenberg structure, meaning non-zero elements are on or below the first superdiagonal. Here is an example of an odd centrosymmetric matrix A,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This matrix is centrosymmetric because it satisfies $A = -JAJ$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It also has a lower Hessenberg structure because non-zero elements are on or below the first superdiagonal. An orthogonal matrix satisfies

$$Q^T Q = I$$

where Q^T is the transpose of Q, and I is the identity matrix. Here is an example of an orthogonal matrix Q

$$Q = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 \end{pmatrix}$$

This matrix is orthogonal because $Q^T Q = I$. We perform the matrix multiplication AQ,

$$AQ = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 \end{pmatrix}$$

The result is

$$AQ = \frac{1}{\sqrt{3}} \begin{pmatrix} 4 & 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & -1 \end{pmatrix}$$

The transpose of matrix A is

$$A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since A is odd centrosymmetric, we know $A^T = -A$. Now we compute $Q^T A$,

$$Q^T A = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The result is

$$Q^T A = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & -2 & 2 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The results above showed the special centrosymmetric matrix has a lower Hessenberg form, as the block matrix possesses its own orthogonal matrix that can be constructed in a block-sparse manner from the centrosymmetric matrix. This process is based on the evaluation of the properties inherent to the special centrosymmetric matrix.

Conclusion

The centrosymmetric matrix possesses a special structural form in its entries, allowing it to be classified based on whether the matrix is of odd or even order. By utilizing an orthogonal matrix, the centrosymmetric matrix can be constructed as a sparse block matrix, in which a lower Hessenberg structure emerges within the blocks. The orthogonal matrices for odd and even orders serve as the fundamental components for forming the block representation of the centrosymmetric matrix. This matrix form results in a sparse structure that is highly beneficial for computational applications. Furthermore, future work can focus on constructing orthogonal matrices corresponding to other types of block matrices derived from centrosymmetric matrices.

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