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Inductive and Intuitive in Proving the Identity Element of a Group: A Structure of Argumentation

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Abstract

This research applies the Toulmin Argumentation Model in identifying the argumentation structure and proving the identity property of a group with an inductive and intuitive approach. This research is qualitative descriptive research, and the instrument is one problem of proving the identity property of a group. Data were collected from a cohort of 30 students who enrolled in the Introduction to Ring Theory course. All participants selected one Subject based on complete and correct answers for further analysis. The results showed that the Subject used his intuition to take one element, which would then be proven as the identity element of a group. Then, the Subject used an Inductive Warrant to prove it. The Subject only takes one element that fulfils the identity property. However, the Subject states that the conclusion applies in general, which makes the argument invalid because valid arguments are based on deductive. At the same time, students at the college level should use a deductive approach to proof. By using the Toulmin Argumentation Model, the subject's ability to formulate and construct arguments in a proof can be seen as structured and clear. This model allows a systematic explanation and strengthens confidence in the validity of the evidence. This research is expected to enrich the understanding of argumentation structure in inductive proofs of group identity properties and show the potential use of the Toulmin Argumentation Model in identifying more in-depth mathematical argumentation.

Keywords: Argumentation Structure; Group Theory; Identity Element; Inductive-Warrant; Intuitive

Abstrak

Penelitian ini menerapkan Model Argumentasi Toulmin dalam mengidentifikasi struktur argumentasi pada pembuktian sifat identitas suatu grup dengan pendekatan induktif dan intuitif. Penelitian ini termasuk penelitian dekriptif kualitatif, dan instrumen berupa satu soal pembuktian sifat identitas suatu grup. Data diperoleh dari 30 mahasiswa yang menempuh mata kuliah Pengantar Teori Gelanggang. Dari seluruh partisipan, satu subjek dipilih berdasarkan jawaban lengkap dan benar untuk dianalisis lebih lanjut. Hasil penelitian menunjukkan bahwa subjek menggunakan intuisinya dalam mengambil satu elemen yang kemudian akan dibuktikan sebagai elemen identitas suatu Grup. Kemudian Subjek menggunakan Warrant Induktif dalam membuktikannya. Subjek hanya mengambil satu elemen yang memenuhi sifat identitas, namun subjek menyatakan bahwa kesimpulan tersebut berlaku secara umum, yang menjadikan argumenya tidak valid. Karena argumen yang valid didasari pada deduktif. Padahal mahasiswa di tingkat perguruan tinggi seharusnya menggunakan pendekatan deduktif dalam pembuktian. Dengan menggunakan Model Argumentasi Toulmin, kemampuan Subjek dalam merumuskan mengonstruksi argumen dalam pembuktian dapat terlihat terstruktur dan jelas. Model ini memungkinkan penjelasan yang sistematis dan memperkuat kepercayaan terhadap validitas pembuktian. Penelitian ini juga menunjukkan kekuatan metode induktif dan intuitif dalam pembuktian matematika, menunjukkan bagaimana pendekatan ini dapat menghasilkan pemahaman yang lebih komprehensif tentang pembuktian matematika. Penelitian ini diharapkan dapat memperkaya pemahaman mengenai struktur argumentasi dalam pembuktian induktif sifat identitas grup dan menunjukkan potensi penggunaan Model Argumentasi Toulmin dalam mengidentifikasi argumentasi matematika yang lebih mendalam.

Kata Kunci: Inductive-Warrant; Intuitif, Elemen Identitas; Struktur Argumentasi; Teori Grup

Introduction

Proof is not a new thing for mathematics students in higher education. Proof is the basis of mathematical understanding and is essential for developing, constructing, and communicating mathematics (Nadlifah & Prabawanto, 2017). Proof is one of the main characteristics of mathematics as a discipline (Rabin & Quarfoot, 2021) and is fundamental in mathematical activities (Hanna, 2018; Wittmann, 2021). The validity of theorems in mathematics can be demonstrated by proof (CadwalladerOlsker, 2011; Pala et al., 2021). Furthermore, proof and reasoning play an important role in showing the correctness of the solution to mathematical problems in mathematics learning (Wittmann, 2021), so the ability to construct evidence for mathematics students is one of the most important things (Moore, 2016; Thomas et al., 2015; Wasserman et al., 2018). Lee (2016) defines proof construction as the process of constructing mathematical statements to show that mathematical propositions are true or false.

Abstract Algebra is a course that necessitates the use of mathematical proof skills (Isnarto et al., 2014). This subject holds significant importance within the mathematics curriculum as it introduces essential concepts such as groups, rings, and fields, which are critical for advanced understanding. Additionally, it enhances abstract thinking and problem-solving abilities (Arnawa et al., 2020). Mastery of these abilities is a crucial aspect of learning Abstract Algebra (Findell, 2001). Given the course's focus on definitions and theorems requiring proof, students must comprehend each definition and theorem thoroughly to effectively organize concepts during proof activities (Arifin et al., 2023; Astuti & Zulhendri, 2017; Pramasdyahsari et al., 2022).

Toulmin stated that identifying the structure of an argument can use the Toulmin Argumentation Model (Toulmin, 2003). The Toulmin model of argumentation offers several advantages over other argumentation models. Arifin et al. (2023) and Andriani et al. (2023) both emphasize its ability to structure and clarify arguments, leading to more systematic explanations and increased confidence in the validity of the arguments. Toulmin's scheme can be used in the analysis, assessment, and construction of arguments, and by using the scheme it can be seen whether the argument is supported by valid data, what guarantees are used to state a valid argument, whether there is a refutation of the argument (Banegas, 2013). The Toulmin Argumentation Model consists of three main components: Data (D), Claim (C), and Warrant (W), and three complementary components: Backing (B), Rebuttal (R), and Qualifier (Q).

According to Toulmin (2003), Data (D) is the foundation of an argument. It consists of facts that support the Claim. Claim (C) is a statement or conclusion made based on data. Warrant (W) is the bridge that connects data and Claims and becomes the rationale or reasoning used to produce the conclusion. A Warrant is strengthened by Backing (B), which is further evidence or additional reasoning that is needed. A Rebuttal (R) is a statement that refutes the resulting conclusion if the condition is not met (Toulmin, 2003). The benefit of analysing an argument with Toulmin's scheme is to capture the best meaning or power of words and propositions by seeing how one can use them in various contexts (Bizup, 2009). In this study, the types of Warrant are classified based on the types of Warrant according to Inglis et al. (2007) , namely inductive, structural-intuitive, and deductive. Tristanti et al. (2017) classified structural-intuitive Warrant and inductive Warrant as non-deductive Warrant. However, the main discussion is the use of non-deductive Warrant, because students at the tertiary level are expected to be able to prove in a deductive way and not use non-deductive Warrant (Tristanti et al., 2016).

Many previous studies discuss argumentation structure. Research by Faizah et al. (2021), Aaidati et al., (2022), and Arifin & Permadi (2023) investigated the argument structure but did not show how the complete argumentation scheme or structure of students works. Then Laamena & Nusantara (2019) examines the argumentation structure that focuses on the Backing component and shows how Toulmin's complete argumentation structure from student answers, however. However, there are still few studies that focus on proving the identity properties of a group using Warrant-Inductive. Therefore, there is a need for research to describe the argumentation structure used by students in constructing evidence and analyzing it using the Toulmin Argumentation Model.

This study aims to describe the argumentation of students in proving the identity properties of a group using Warrant-Inductive based on the Toulmin Argumentation Model by analyzing and understanding the argumentation components used by students. The results of this study are expected to provide insight for teachers in identifying the components of argumentation used by students. By understanding these components, teachers can be more effective in guiding students in constructing valid evidence. The results of this study are expected to be a guideline for teachers in guiding students in constructing valid evidence because evidence based on Warrant-Inductive is not valid evidence. The findings of this study can be used as a source of ideas to reconstruct students' invalid argumentation so that teachers can provide more specific guidance and help students improve the quality of their argumentation.

Method

This study used a qualitative descriptive research design to describe the structure of students' arguments in constructing proofs on Group Theory material, especially on the closure property. A total of 30 fourth-semester students who were taking the Introduction to Group Theory course at the Department of Mathematics, State University of Malang became research participants. Data were obtained from the answers to the Mathematical Proof Test questions contained in Figure 1 and the results of interviews with students.

Perhatikan himpunan G berikut.

$$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \middle| \; a, b \in \mathbb{R}, b \neq 0 \right\}$$

Buktikan bahwa G adalah grup terhadap operasi kali!

Figure 1. Mathematical Proof Test Questions

The test question consisted of 1 problem proving G is a group. To prove a system $(G,\times)$ is a group, it was necessary to show several properties, namely the operation is closed, the operation is associative, there is an identity element, and every element in G has an inverse. Based on these properties, one property was chosen, namely the closed property which will be further investigated regarding the argumentation constructed by students. Since the closure property is the initial property that must be proven to establish that a set is a group, students must be able to prove it correctly before moving on to the next group property. The problem was validated by an expert in the field of mathematics. Student responses were collected in written form. Then an interview was conducted to explore further information about the argumentation used by the subject when showing the nature of closure. The data from the subject's work and interview transcripts would be analyzed for the argumentation structure based on the Toulmin Argumentation Model by identifying each argumentation component and the relationship between these components.

Results

Figure 2. below is the result of the Subject's work in showing the existence of the identity element in $(G,\times)$.

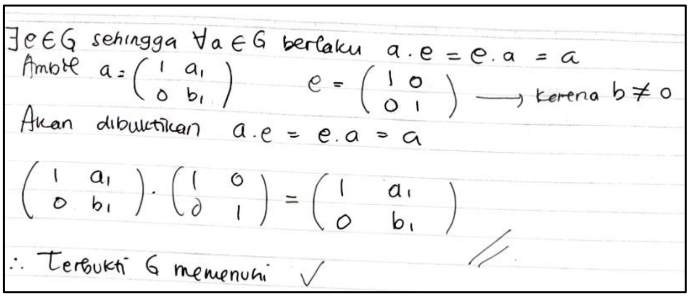

Figure 2. Subject Answer Result

The subject used an inductive approach to show the existence of an identity element in $(G,\times)$. The Data Component is the known set G with multiplication operation, with the Claim that there is an identity element in $(G,\times)$ based on the Warrant of identity definition, namely, there is an element $e \in G$ so that $\forall a \in G$ holds $a \cdot e = e \cdot a = a$. The Warrant used by the Subject is correct because it is by the definition of identity. Then an interview was conducted to clarify the results of the work. (Note: R as Researcher and S as Subject)

R : *"What do you want to show in this answer to number 3?"*

S : *"So in the group, there is an identity element, If I take element "a" in G, then multiplication the identity element is equal to the identity element times a and is equal to itself."*

R : *"So what is the definition of the identity element?"*

S : *"So there is an element* $e \in G$ *such that* $\forall a \in G$ *holds* $a \cdot$

$e = e \cdot a = a$"

Based on the results of the work and the interview, it is known that the Subject conducted the proof with an inductive approach starting by assuming that the system $(G,\times)$ has an identity element e and postulating the identity element e is $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which then becomes the Claim component. Then proving it based on the Warrant of the identity definition and the reason that supports the Warrant becomes Backing, which is the result of matrix multiplication. To find out why the Subject took the matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to be proved as the identity element in $(G,\times)$, an interview was conducted below

R : *"Then you wrote take $a = \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$ $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$because $b \neq 0$. Can you explain what that step means?"*

S : *"I want to prove $a \cdot e = e \cdot a = a$, so I took the identity element which is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then I took another element of G which is matrix a which is $\begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$ because $a_1, b_1 \in R$ and $b_1 \neq 0$ then I multiplied the two"*

R : *"Why do you believe the identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, not something else?"*

S : *"Because as far as I understand the identity element of the matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$"*

Based on the interview transcript above, the Subject will prove that there is an identity element by taking the matrix $a = \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$ and then operating with $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then it will be shown that $a \cdot e = e \cdot a = a$. However, in the work, the Subject only calculated the result of $a \cdot e = a$, and did not write the result of $e \cdot a = a$. Therefore, an interview was conducted as below to clarify the Subject's work.

R : *"Then you wrote the multiplication $\begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$. What do you want to show from that step, isn't there only one equation?"*

S : *"Yes, that's missing, it only shows a multiplied by the identity, but there (the work) I didn't show the identity multiplied by a, so I missed it yesterday, but there I wanted to prove $ae = a$, and actually wanted to prove* *that $ea = a$ too but I didn't write it there"*

The Subject realized that he only calculated the result of $a \cdot e = a$, not writing down the result of $e \cdot a = a$. Then similarly, the Subject could not give a reason (Warrant) why he took the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to be proved as the identity. The subject used his understanding that the identity on the matrix with multiplication operation is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. To achieve a valid proof with an inductive approach, it cannot be a single case and then generalized, there must be other guarantors so that the proof with the inductive approach becomes valid. The subject did not know the Warrant that there was no other matrix besides $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as the identity is because the identity in a group is single so the subject immediately takes one matrix which is then proven that the matrix is the identity on $(G,\times)$. The subject expressed uncertainty (Qualifier) regarding the results of their work due to having considered only one example and lacking knowledge on how to identify identity elements in other groups. The structure of the Subject's argumentation is presented in Figure 3 below.

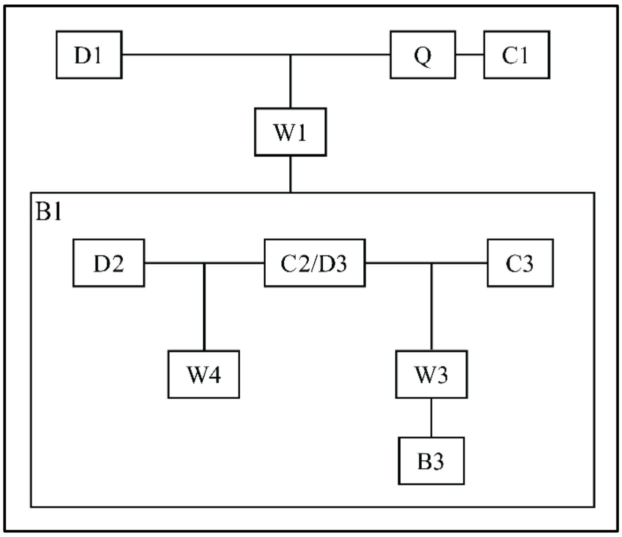

Figure 3. Subject's argumentation structure in proving the identity property

The description of Figure 3 above is contained in Table 1 below.

Table 1. Description of Subject Argumentation Structure

Code	Statement
	Argument 1
D1	$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}, b \neq 0 \right\}$ with multiplication
Q	Sure
C1	$(G,\times)$ have an identity element
W1	Exist $e \in G$ that satisfied $a \cdot e = e \cdot a = a$ for all $a \in G$
B1	Argument 2 and argument 3
	Argument 2
D2	$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}, b \neq 0 \right\}$ and $a = \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$
C2	$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$
W2	Definition of element of G
	Argument 3
D3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$
C3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity
W3	Definition of Identity element
B3	$\begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$

Discussion

Based on the data exposure of the research results, it is known that the Subject starts proving the identity properties by using accurate data based on the information obtained on the problem. According to Faizah et al. (2018), accurate and complete data is strong evidence to support the claim. Subjects and Subjects did not write complete data, but in the interview process, both subjects were able to mention the data accurately. Both subjects were also able to mention the definition of the identity element correctly, where it acts as a Warrant component or guarantor of the Claim that there is an identity element in $(G,\times)$. The subject's understanding of the Warrant used is very important because the Warrant is a guarantor used to determine the truth of the conclusion of an argument (Faizah et al., 2020a).

Then the Subject shows that the Warrant applies to $(G,\times)$ and the work becomes the Backing component because Backing is further evidence or additional reasons needed (Laamena & Nusantara, 2019). Based on the results of the subject's work, this Backing component consists of several more sub-arguments that show that there is an identity element in $(G,\times)$, Knipping & Reid (2019) call it the term local argument, which is steps of the argument, which allows the subject to choose different arguments in the proof process.

In contrast to the proof of the previous properties, both subjects proved that the Warrant applies to $(G,\times)$ with an inductive approach, which is taking one special case example to be proven. Both subjects took one matrix, namely matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$ which will be proved to fulfill $ae = ea = a$ for $a \in G$. An argument with an inductive approach is an invalid argument even though the conclusion given is true (Tristanti et al., 2017). The subject felt confident (Qualifier) with the answer given despite realizing that there was a calculation error and the Rebuttal component did not appear in the proof of the identity property because Rebuttal does not always appear depending on the statement to be proven whether it requires a counterexample or not (Lin, 2018).

In this study, the types of Warrant produced by the two subjects were classified based on the types of Warrant, namely inductive, structural-intuitive, and deductive (Inglis et al., 2007). Tristanti classified structural-intuitive Warrant and inductive Warrant as non-deductive Warrant (Tristanti et al., 2017). However, the main discussion is the use of non-deductive Warrant, because students at the tertiary level are expected to be able to prove in a deductive way and not use non-deductive Warrant (Tristanti et al., 2016).

Based on the results of the Subject's work on proving the identity property, it was found that the subject used his intuitive knowledge. Intuitive knowledge is characterized by its concise and straightforward nature. It is related to personal feelings about evidence, which is a person's personal feeling that concepts are clear and that properties and statements are intrinsically true (Antonini, 2019). Based on intuition from previous learning experiences that the identity matrix is the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then find the inverse matrix with the matrix inverse formula, and the multiplication of two matrices is not commutative. By using inductive proof to show his intuition is correct in proving the identity property, namely by taking the matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which will be proven to fulfil $ae = ea = a$ for all $a \in G$. With such a method of proof, it can be said that it is true that there is an identity element in (G,×), but only for that one special case, namely, only for the case of matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, there is no guarantee that there is no other matrix that satisfies $ae = ea = a$ besides matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Both subjects also used the matrix inverse formula to determine the inverse element (a^{-1}) of an on (G,×) and proved that the matrix inverse satisfies $aa^{-1} = a^{-1}a = e$ for all $a \in G$. The same thing was also done by the Subject in proving the abelian group. Based on his intuition, the subject guessed that the multiplication of two matrices is not commutative. Although using inductive Warrant, of course, the subject must understand that Claim cannot be generalized so that the truth of the conclusion only applies to the specific case being tested (Inglis et al., 2007). Subjects do not realize that they are constructing an invalid argument, because valid arguments must be based on deduction (Imamoglu & Togrol, 2015).

The use of structural-intuitive and inductive Warrant is an important concern in mathematical argumentation (Inglis et al., 2007). This is based on thinking using structural-intuitive Warrant that generates conjectures. When a person believes that his conjecture does not need to be proven he is thinking intuitively. But someone thinks that his conjecture is believed to be true and he also thinks that his conjecture must be proven deductively to be more confident, one way of validating the conjecture by deduction (Tristanti et al., 2015). The non-deductive Warrant type is used to reduce the uncertainty of the conclusion expressed by the subject. In addition, the subject uses the deductive Warrant type to eliminate the uncertainty of the conclusion, so that the conclusion obtained from the deductive Warrant is certain (Inglis et al., 2007).

According to Tall's "Three Worlds of Mathematics" Theory, mathematical proof in higher education falls within the axiomatic-formal world. This contrasts with school-level mathematics, which belongs to the conceptual-embodied world based on perception and reflection, and the proceptual-symbolic world, where actions like counting evolve into symbolic concepts like numbers (Tall, 2008). Based on this theory, students at the tertiary level should be able to perform deductive proofs, although it starts with using intuition and induction, which then considers many cases and examines patterns and relationships to produce generalizations. Once the generalization is reached, its truth must be established using the deductive process. Then students should be able to complete mathematical proofs related to abstract Algebra correctly following Piaget (1964) developmental stages which state that a person can carry out formal or abstract thinking processes after the age of 12 years and over (Faizah et al., 2020b).

Future research is expected to extend the analysis by applying Toulmin's Argumentation Model to other concepts in group theory or other mathematics branches, not only to prove problems. In addition, inductive and intuitive proofs are expected to be further analyzed to understand how these approaches can influence how students learn and understand complex mathematical concepts.

Conclusion

The subject's understanding of proving a group's identity properties can be illustrated from the constructed argumentation, which is then identified with the Toulmin Argumentation Model. The subject can use the Data correctly, propose a valid value claim, and provide reasons for the claim. However, the subject proved this with an inductive approach. Whereas students at the tertiary level must be able to prove deductively, even though it begins with intuition and induction, the truth must be determined using deductive. This research is limited to subjects who provide inductive proof of the element of identity. More research needs to be done to determine how students prove by deductive proof and on other group properties such as closure, associative, and inverse properties. Knowing the proof structure constructed by students can be a consideration for lecturers to design lessons that facilitate students to improve their argumentation skills in proving group properties

The findings of this study have valuable implications for both students and lecturers. Regarding learning methods and curriculum development, lecturers can design more effective learning methods by integrating inductive and intuitive approaches to proving a statement, especially on mathematical proof at the college level. This will help improve understanding of abstract concepts, improve logical thinking, encourage the development of problem-solving skills, and improve proof ability. In addition, the findings of this study can be used by lecturers to create teaching materials by including examples of diverse and more easily understood proofs.

This research can also help students make valid arguments to prove problems in abstract algebra material, which is crucial to developing their critical and analytical thinking skills. Students need to understand the basics of theories, such as groups, and be able to use relevant definitions and theorems appropriately. Valid arguments require rigorous mathematical logic, examples and counterexamples, as well as clear and structured writing. Students can then use a methodical and logical approach to prove their statements.

This study has several limitations that need to be considered. Firstly, the generalisability of the findings may be limited as the inductive and intuitive methods tested in the context of identity elements in groups may only be directly applicable to other mathematical concepts with adjustment. Each mathematical concept has unique characteristics that may require different proof strategies. In addition, if this study was conducted on a limited sample, the results may not reflect the wider population, as variability in students' educational backgrounds, mathematical abilities, and learning styles may impact the effectiveness of these approaches.

Secondly, inductive and intuitive approaches may be more effective for straightforward or intermediate mathematical concepts. There may need to be more than this approach for highly complex and contract concepts with the support of more rigorous formal methods. In addition, intuition in mathematical proof is subjective and can vary between individuals, posing challenges in designing appropriate teaching materials for all students. Limitations of the research methodology, such as potential biases in observations or interviews, may also affect the validity and reliability of the findings, which need to be considered in interpreting the results of this study.

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